Optimal Estimation of Rain Rate Profiles from Single-Frequency Radar Echoes

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ABSTRACT

It is well-documented (Hitschfeld and Bordan 1954, Meneghini 1978, Haddad et al 1993) that there are significant ambiguities inherent in the determination of a particular vertical rain intensity profile from a given time profile of radar echo powers measured by a downward- looking (spaceborne or airborne) radar at a single attenuating frequency. Indeed, one already knows (Haddad et al 1993) how to vary the parameters of the reflectivity- rainrate ($Z-R$) and attenuation- rainrate ($k-R$) relationships in order to produce several substantially different rain rate profiles which would produce the same radar power profile. Imposing the additional constraint that the path-averaged rain- rate be a given fixed number does reduce the ambiguities but falls far short of eliminating them (Haddad et al 1994). While we have derived the formulas to generate all *deterministic* mutually ambiguous rain rate profiles from a given profile of received radar reflectivities, there remains to produce a quantitative measure to assess how likely each of these deterministic profiles is, what the appropriate "average" profile should be, and what the "variance" of these multiple solutions is. Of course, in order to do this, one needs to spell out the stochastic constraints that can allow us to make sense of the words "average" and "variance" in a mathematically rigorous way. Such a quantitative approach would be particularly well-suited for such systems as the spaceborne Ku-band Precipitation Radar of the Tropical Rainfall Measuring Mission (TRMM). Indeed, one would then be able to use the radar reflectivities measured by the TRMM radar to estimate the rain rate profile that would most likely have produced the measurements, as well as the uncertainty in the estimated rain rates, as a function of range. This paper presents an optimal approach to solve this problem.

MATHEMATICAL APPROACH

For simplicity, we start with the model that the effective reflectivity $p(r)$, measured at range r by a downward-looking monostatic narrow-band radar such as the TRMM Precipitation Radar, is proportional to the reflectivity coefficient Z of the rain at range r, and to the accumulated attenuation from range O (the top of the cloud) to range r. Calling k(r) (resp. $R(r)$) the attenuation coefficient (resp. rain rate) at range r, we assume for simplicity that $Z = aR^b$ and $k = \alpha R^\beta$ for some value of the parameters a, b, α and β, and that the calibrated reflectivity is therefore given by

$$p(r) = aR(r)^b 10^{-0.1(2\int_0^r \alpha R(t)^\beta dt)} \qquad (1)$$

Treating a, b, α and β as parameters, the solution to equation (1) can be written as

$$R(r) = \frac{p(r)^{1/b}}{\left(a^{\beta/b} - \frac{0.2\log(10)\alpha\beta}{b}\int_0^r p(t)^{\beta/b}dt\right)^{1/\beta}} \qquad (2)$$

Equation (2) suggests that if the rain parameters me not known exactly, multiple solutions for R can exits. In (Haddad et al 1993), we describe just how mutually ambiguous these multiple solutions can get. In the same paper, we also show that using the surface return as a reference does not solve the ambiguity problem. Since one has to 'live with" these ambiguities, it is very important to know how likely each of the multiple solutions is: specifically, given some a-priori "statistical" constraints on the variables involved, one would like to find what the "average" solution to (2). Using average values for the rain parameters is still not sufficient because even when exact values for a, b, α and β are given, it is known that the numerical implementation of equation (2) gives a numerically *unstable* "inversion" algorithm].

Thus onc is naturally led to a stochastic filtering approach. One would like to introduce a "measure" on the set of all ambiguous profiles giving rise to the same measured reflectivity profile, and try to find the "average" profile with respect to this measure on this set, along with an estimate of the mean difference between the members of this set of mutually ambiguous profiles. In (Haddad et al, 1993), we described an algorithm to compute the joint probability density function 'P for {If(r), a, b, α, β} given measurements of *p(r)*. The 'average" rain profile and the "mean deviation" with all the mutually ambiguous profiles can then be obtained from the moments of 'P. Indeed, the results reported in (Haddad et al, 1993) have been very encouraging. In particular, in the case where *a, b*, α and β are assumed known, *this approach yields a* stable inversion *algorithm which does not require any surface reference information.* But calculating the full density function requires large amounts of computer memory and CPU time, too large to make the algorithm useful in anywhere near real-time. In *order* to reduce the amount of computer resources required, rather than calculating $\mathcal{P}$ itself, one can try to compute its mean and covariance directly. This amounts to deriving the extended Kalman filter appropriate to the problem at hand. We now describe how this is done.

First, we need to specify the a-priori constraints on tbe "state variables" R(r), a, b, α, β and c(r) $= \int_0^r \alpha R^\beta$. For simplicity, we shall assume that a, b, α and β are constant, that the only constraint on c is that it be the integral with respect to r of $\alpha R(r)^\beta$, and we express the requirement that R itself be positive and continuous by writing

$$R(r) = e^{x(r)+\lambda r} \qquad (3)$$

where x is the (mathematically) simplest continuous stochastic process and λ a suitable factor (possibly zero) to be determined. Specifically, without further a-priori information, wc assume that $x(r) =$ z(O) + *ah(r)*, where $x(0)$ and b(r) are independent, $x(0)$ itself is Gaussian with mean m_0 and 'variance σ_0^2, and the process $b(r)$ has independent 0-mean Gaussian increments with variance equal to the extent in range of the increment interval. '1'bus, in effect, we are assuming that the a-priori constraints on the evolution of $\log(R)$ with range r are those of standard Brownian motion, up to a possible 'drift'' term λr.

Now that wc have established the a-priori constraints on the dynamics of our variables, we must make explicit the function h(r) expressing our measurement from range r in terms of our state variables. From equation (l), one can see that

$$h(r) = log(a) + b(x(r) + \lambda r) - 0.2\log(10)c(r) + \text{Noise}. \qquad (4)$$

Let us write σ_N for the r.m.s. noise level in the measurements, which, for simplicity, we shall attribute here to Rayleigh fading only (system noise can be taken into account, at the expense of making the exposition somewhat more cumbersome). Since our data consist of the averaged power of M independent pulses, the noise term in (4) would be the logarithm of the average of the squared-magnitudes of M independent standard complex Gaussian variables. Hence, as soon as $M > 4$, it is quite reasonable to assume that this noise term is itself approximately a O mean normal variable with variance $\sigma_N^2 \simeq 1/M$.

We are now ready to apply the standard machinery of stochastic filtering to obtain the best estimate $\hat{R}(r)$ of the rain rate at range r given all the observations. Since the relation $dc/dr = \alpha R^\beta$ is non-linear, we cannot usc a straightforward Kalman filter to solve the problem. Wc chose to use an extended Kalman filter approach, using a first-order Taylor series linearization to obtain both the forward estimate (starting from the top of the cloud r = O) and the backward estimate (starting from the ocean surface). The theory and details behind the technique can be found for example in (Øksendal, 1985). For completeness, we summarize the flow of the particular algorithm in the case at hand, when the parameters a, b, α and β are assumed known. First, one must obtain estimates $\hat{x}(r)$ and $\hat{c}(r)$ of the state variables x and c at all ranges r based on all earlier measurements obtained for r' < r, along with their covariances $p_{xx}(r)$, *pa(r)* and $p_{cc}(r)$. To do this, one must start with

$$\hat{x}(0) = m_0 \qquad (5)$$

$$\hat{c}(0) = 0 \qquad (6)$$

$$p_{cc}(0) = 0 \qquad (7)$$

$$p_{cx}(0) = 0 \qquad (8)$$

$$p_{xx}(0) = \sigma_0^2. \qquad (9)$$

Then, given our estimates at range r, the estimates at range r + 6 can be obtained in two steps, by first accounting for the changes in the dynamics using the formulas

$$\tilde{x}(r + \delta) = \hat{x}(r) \qquad (10)$$

$$\tilde{c}(r + \delta) = \hat{c}(r) + \alpha \int_r^{r+\delta} e^{\beta(x(t)+\lambda t)}dt \qquad (11)$$

$$\tilde{p}_{xx}(r + \delta) = p_{xx}(r) + \sigma^2 6 \qquad (12)$$

$$\tilde{p}_{cx}(r + \delta) = p_{cx}(r) + \alpha\beta \int_r^{r+\delta} e^{\beta(x(t)+\lambda t)}\tilde{p}_{xx}(t)dt \qquad (13)$$

$$\tilde{p}_{cc}(r + 6) = p_{cc}(r) + 2\alpha\beta \int_r^{r+\delta} e^{\beta(x(t)+\lambda t)}\tilde{p}_{cx}(t)dt, \qquad (14)$$

then by accounting for the measurement $z(r+\delta)$ obtained from range r + δ using the formulas

$$\hat{x}(r + \delta) = \tilde{x}(r + \delta) + \frac{b\tilde{p}_{xx}(r+\delta) - 0.2\log(10)\tilde{p}_{cx}}{D}\Delta \qquad (15)$$

$$\hat{c}(r + \delta) = \tilde{c}(r + \delta) + \frac{b\tilde{p}_{cx}(r+\delta) - 0.2\log(10)\tilde{p}_{cc}}{D}\Delta \qquad (16)$$

where $D = (0.2\log(10))^2\tilde{p}_{cc}(r+\delta) - 2(0.2\log(10))b\tilde{p}_{cx}(r + 6) + b^2\tilde{p}_{xx}(r+\delta) + \sigma_N^2$, and $\Delta = z(r+6) - (\log(a) + b(\tilde{x}(r + c\$) + \lambda r) - 0.2\log(10)\tilde{c}(r+\delta))$. The backward portion of the algorithm is similar, except for the obvious sign changes that are then necessary. The resulting procedure is orders of magnitude more efficient than the full density function approach described in (Haddad et al, 1993). Its extension the the case of coupled Z-R and k-R relations discussed in (Haddad et al, 1994) is straightforward.

APPLICATIONS

Before we can describe practical applications, wc still need to discuss the choice one must make for the parameters m_0, σ_0, σ and λ. Although in practice it turns out that the exact values do not affect the estimation algorithm significantly (after all, when a, b, α and β arc known, the theoretical solution is unique), one should certainly make an effort to give them physically reasonable and realistic. values. To do that, we usc the a-priori constraints which we have imposed. It follows from (3) that the expected value of the rain-rate R(0) at the top of the rain column is

$$E\{R(0)\} = e^{m_0+\frac{1}{2}\sigma_0^2} \qquad (17)$$

and its relative variance is

$$\mathcal{E}\left\{\left(\frac{R(0)}{\mathcal{E}\{R(0)\}} - 1\right)^2\right\} = e^{\sigma_0^2} - 1 \qquad (18)$$

In practice, we set a minimum "threshold onset" value R_{min} for the smallest significant rain rate we expect at the top of the rain column, along with some estimate for the associated mean relative uncertainty. Equation (18) then implies that we should choose

$$\sigma_0^2 = \log\left(1 + \mathcal{E}\left\{\left(\frac{R(0)}{R_{min}} - 1\right)^2\right\}\right) \qquad (19)$$

and (17) in turn implies that we should then choose

$$m_0 = \log(R_{min}) - \frac{1}{2}\sigma_0^2 \qquad (20)$$

The choice of λ is somewhat more problematic. We do know that, a priori, by definition, the rain rate should initially increase with range from the a-priori value R_{min}. This would imply a positive drift λ. To get a value for λ, wc look at the terminal behavior of R. Writing R'(r) for $R'(r) = R(r_s - r)$, where r_s is the range of the surface, and if we reverse the constraint (3) in time to apply it to R'(r), one finds that the "a-priori" (with time reversed) expected value for R' is given by

$$\mathcal{E}\{R'(r)\} = e^{m_{r_s}+\frac{1}{2}\sigma_{r_s}^2+\lambda r_s} \cdot e^{(\frac{1}{2}\sigma^2-\lambda)r} \qquad (21)$$

Since we have a priori no reason to expect the rain rate to increase or decrease as one moves up from the bottom of the rain column, it is natural to choose the value

$$\lambda = \frac{1}{2}\sigma^2 \qquad (22)$$

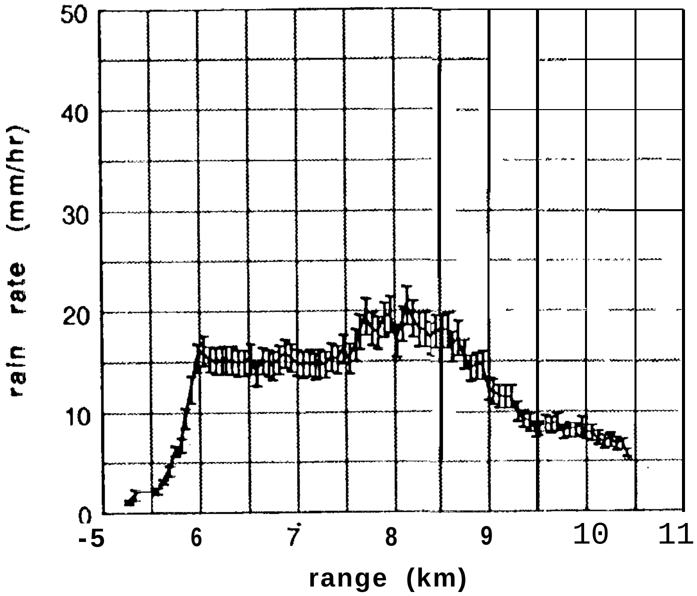
Last, we must decide on a value for σ. In practice, we would expect the average rain rate R_{avg} over the rain column to be greater than the minimum value R_{min} at the top of the column (otherwise our data is of little interest). From equation (3) and our choice of λ, one can verify that

$$\frac{1}{r_s}\int_0^{r_s} \mathcal{E}\{R(r)\}dr = \frac{e^{\sigma^2 r_s} - 1}{\sigma^2 r_s} R_{min} \qquad (23)$$

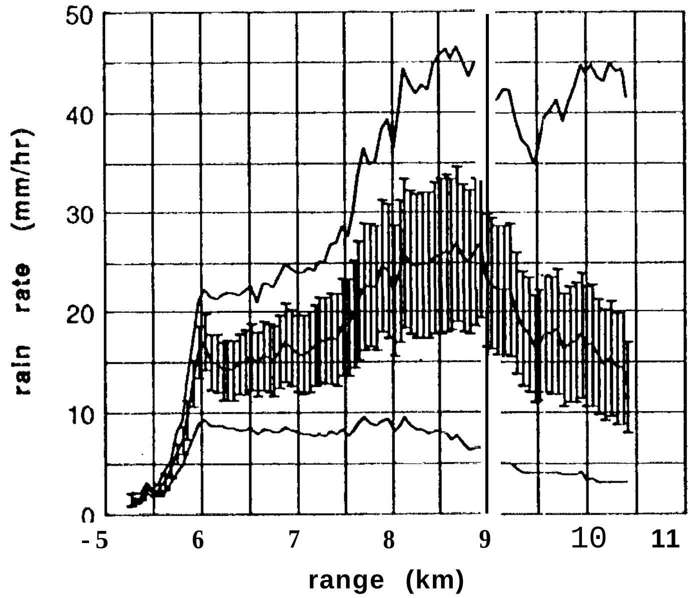
, It is therefore natural to choose σ by making this quantity equal to an expected average value R_{avg}.

The graph to the right shows the estimated rain rate obtained using the algorithm described above, when the input was one of the radar reflectivity profiles measured by JPL's ARMAR radar (Durden et al, 1992) over the Western Pacific Ocean during the TOGA-COARE experiment in February 1993. Details of the participation of ARMAR in COARE can be found in (Li et al, 1993). We used the values $a = 300$, $b = 1.4$, $\alpha = 0.026$ and β , 1.11, along with $R_{min} = 1$ mm/hr ($\pm$ 50%), and $R_{avg} = 10$ mm/hr. For comparison, the graph below reproduces the estimates obtained using our full-density-functiorl code (from Haddad et al, 1993), where we had assumed that $200 < a < 400$, $1.4 < b < 1.6$, $0.018 < \alpha < 0.034$ and $\beta = 1.08$. A more comprehensive analysis of the algorithm described above will be ready shortly.

Feb 4, 15:35 - Convective rain
(Mode D - Extended K)



Feb 4, 15:35 - Convective rain
(Mode D)



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